Asymmetric Coupled Transmission Lines with Anisotropic Coupling

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Abstract — The characteristics of asymmetric coupled transmission lines are derived for the case of general anisotropic coupling special cases of which are certain geometries of anisotropic materials or from distributed active devices. In addition to deriving the terminal characteristics for the uniform coupled-line four-port case with anisotropic coupling, it is shown that alternate equivalent expressions for the mode impedances and admittances can be derived which have a simplified form. These simplified expressions are useful when examining special cases where a good approximation can be achieved by considering either a generalized anisotropic mutual impedance or a generalized anisotropic mutual admittance.

I. INTRODUCTION

Coupled line topologies are employed in a wide range of microwave and millimeter-wave circuits to provide signal routing, and perform frequency selective functions. Previous analyses have dealt with geometries where the media is inhomogeneous and the coupled lines are asymmetric [1-5] but not where the coupling is of a general anisotropic form. The general anisotropic case includes geometries involving nonreciprocal material such as ferrites and also addresses the situation of distributed active devices [6,7] such as field effect or junction transistors.

In this paper a general asymmetric coupled line with anisotropic coupling is studied and expressions are derived for the propagation constants and characteristic admittances and impedances in terms of the voltages and currents on the lines. It is shown that expressions for the line impedances and admittances can be formulated, which, in contrast to previously published results, not only describe the more general anisotropic coupling case but have a simplified algebraic form. Indeed, in certain special cases, these expressions can be further simplified and lead to a more intuitive description of the characteristic coupled-line parameters. Similar to the case of isotropic coupling, the modes correspond to linear combinations of the voltages and currents on the lines. Unlike the case where the coupling is bilateral, the four-port admittance matrix no longer exhibits the symmetry of the bilateral-coupling case.

II. COUPLED-LINE ANALYSIS

The circuit description of the coupled lines assumes that the differential voltages and currents along the transmission line are linear functions of the currents and voltages, respectively, along both lines. Thus, it is assumed that the relationships between the currents, i_1 and i_2, and the voltages, v_1 and v_2, for two coupled transmission lines can be written as:

$$-\frac{dv_1}{dx} = z_{11}i_1 + z_{12}i_2 \qquad -\frac{dv_2}{dx} = z_{21}i_1 + z_{22}i_2 \tag{1}$$

$$-\frac{di_1}{dx} = y_{11}v_1 + y_{12}v_2 \qquad -\frac{di_2}{dx} = y_{21}v_1 + y_{22}v_2 \tag{2}$$

where $z_{ii}\,(i = 1, 2)$ and $y_{ii}\,(i = 1, 2)$ are the self-impedances and self-admittances per unit length of line i in the presence of line j $(j = 1, 2; j \neq i)$. Similarly, $z_{ij}\,(ij = 12, 21)$ and $y_{ij}\,(ij = 12, 21)$ are the mutual impedances and mutual admittances per unit length, respectively, between the lines.

This description allows for non reciprocal coupling between the lines, i.e. energy can transfer from line *1* to line *2* differently than from line 2 to line *1*. Hence, this formulation not only allows for certain orientations of nonreciprocal material such as ferrites, but also may be applied to situations where a distributed active device provides coupling of the two transmission lines. However, this does not allow for the most general case of non reciprocal behavior since a wave traveling in the $+x$ direction is assumed to behave the same as one traveling in the $-x$ direction. An $e^{j\omega t}$ time dependence and an x dependence of the voltages and currents given by $e^{\pm\gamma x}$ is assumed.

A. Coupled Voltage Equations

After differentiating (1) and substituting with (2) the following set of coupled voltage equations results:

$$(\gamma^2 - a_{11})v_1 + a_{12}v_2 = 0 \qquad a_{21}v_1 + (\gamma^2 - a_{22})v_2 = 0 \tag{3}$$

where

$$a_{ij} = z_{i1}y_{1j} + z_{i2}y_{2j} \tag{4}$$

Thus, the four roots of the characteristic equation are given by:

$$\gamma_{c,\pi}^2 = \frac{1}{2}\left[a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\right] \tag{5}$$

where the modes given by the propagation constant $\pm\gamma_c$ correspond to the forward and backward traveling in-phase mode. Similarly, the modes given by the propagation constant $\pm\gamma_\pi$ correspond to the antiphase forward and backward traveling waves. The ratio of the voltages on the two lines can be found from (3) and is defined by:

$$R_{v_{c,\pi}} = \left.\frac{v_2}{v_1}\right|_{c,\pi} = \frac{(\gamma_{c,\pi}^2 - a_{11})}{a_{12}} = \frac{a_{21}}{(\gamma_{c,\pi}^2 - a_{22})} \tag{6}$$

In the general case, all four waves will exist. Thus the voltages on lines *1* and *2* may be written as:

$$v_1 = A_1 e^{-\gamma_c x} + A_2 e^{\gamma_c x} + A_3 e^{-\gamma_\pi x} + A_4 e^{\gamma_\pi x} \tag{7a}$$

$$v_2 = A_1 R_{v_c} e^{-\gamma_c x} + A_2 R_{v_c} e^{\gamma_c x} + A_3 R_{v_\pi} e^{-\gamma_\pi x} + A_4 R_{v_\pi} e^{\gamma_\pi x} \tag{7b}$$

By substituting (7a) and (7b) into either (1) or (2) the general forms for the currents are given by:

$$i_1 = A_1 Y_{1_c} e^{-\gamma_c x} - A_2 Y_{1_c} e^{\gamma_c x} + A_3 Y_{1_\pi} e^{-\gamma_\pi x} - A_4 Y_{1_\pi} e^{\gamma_\pi x} \quad (8a)$$

$$i_2 = A_1 R_{v_c} Y_{2_c} e^{-\gamma_c x} - A_2 R_{v_c} Y_{2_c} e^{\gamma_c x} + A_3 R_{v_\pi} Y_{2_\pi} e^{-\gamma_\pi x} - A_4 R_{v_\pi} Y_{2_\pi} e^{\gamma_\pi x} \quad (8b)$$

Y_{1_c} and Y_{2_c} are the characteristic admittances for the in-phase modes on lines *1* and *2*, respectively, while Y_{1_π} and Y_{2_π} are the characteristic admittances for the anti-phase modes on lines *1* and *2*, respectively. In the case where (7a) and (7b) are substituted into (1), the admittances are given by:

$$Y_{1_{c,\pi}} = \frac{\gamma_{c,\pi}\left(z_{22} - z_{12} R_{v_{c,\pi}}\right)}{z_{11} z_{22} - z_{12} z_{21}} \qquad Y_{2_{c,\pi}} = \frac{\gamma_{c,\pi}\left(z_{11} - z_{21} R^{-1}_{v_{c,\pi}}\right)}{z_{11} z_{22} - z_{12} z_{21}} \quad (9)$$

In the special case where the coupling is bilateral (9) reduces to the expressions derived by Tripathi.

If (7a) and (7b) are substituted into (2), then the expressions for the admittances are given by:

$$Y_{1_{c,\pi}} = \frac{y_{11} + y_{12} R_{v_{c,\pi}}}{\gamma_{c,\pi}} \qquad Y_{2_{c,\pi}} = \frac{y_{22} + y_{21} R^{-1}_{v_{c,\pi}}}{\gamma_{c,\pi}} \quad (10)$$

Obviously these expressions for the characteristic admittances must be equivalent. This can be shown through straightforward algebraic manipulation. It should be noted, however that (10) are somewhat simpler expressions than (9), and convey more physical insight into the dependence of the coupling on the line parameters.

B. Coupled Current Equations

In a manner analogous to that presented in the above paragraph, an alternate approach is to differentiate (2) and substitute with (1) to obtain the following set of coupled current equations:

$$\left(\gamma^2 - b_{11}\right) i_1 + b_{12} i_2 = 0 \qquad b_{21} i_1 + \left(\gamma^2 - b_{22}\right) i_2 = 0 \quad (11)$$

where

$$b_{ij} = z_{1j} y_{i1} + z_{2j} y_{i2} \quad (12)$$

Thus, the four roots of the characteristic equation are given by:

$$\gamma^2_{c,\pi} = \frac{1}{2}\left[b_{11} + b_{22} \pm \sqrt{(b_{11} - b_{22})^2 + 4 b_{12} b_{21}}\right] \quad (13)$$

Although $a_{ij} \neq b_{ij}$, the dispersion equation given by (13) is identical to that given by (5). Indeed, this must be the case, since the voltages and currents are just components of the same electromagnetic wave. In a manner analogous to the above definition of a voltage ratio, it is appropriate, in this case to define the ratio between the currents flowing in line *1* and line *2*. The ratio of the currents on the two lines can be found from (11) and is defined by:

$$R_{i_{c,\pi}} = \left.\frac{i_2}{i_1}\right|_{c,\pi} = \frac{(\gamma^2_{c,\pi} - b_{11})}{b_{12}} = \frac{b_{21}}{(\gamma^2_{c,\pi} - b_{22})} \quad (14)$$

Just as above, in the general case, all four waves will exist. However, in the following the general form of the currents are considered first. The currents on lines *1* and *2* may be written as:

$$i_1 = A_1 e^{-\gamma_c x} + A_2 e^{\gamma_c x} + A_3 e^{-\gamma_\pi x} + A_4 e^{\gamma_\pi x} \quad (15a)$$

$$i_2 = A_1 R_{i_c} e^{-\gamma_c x} + A_2 R_{i_c} e^{\gamma_c x} + A_3 R_{i_\pi} e^{-\gamma_\pi x} + A_4 R_{i_\pi} e^{\gamma_\pi x} \quad (15b)$$

In an analogous manner, by substituting (15a) and (15b) into either (1) or (2) the general forms for the voltages on the lines are given by:

$$v_1 = A_1 Z_{1_c} e^{-\gamma_c x} - A_2 Z_{1_c} e^{\gamma_c x} + A_3 Z_{1_\pi} e^{-\gamma_\pi x} - A_4 Z_{1_\pi} e^{\gamma_\pi x} \quad (16a)$$

$$v_2 = A_1 R_{i_c} Z_{2_c} e^{-\gamma_c x} - A_2 R_{i_c} Z_{2_c} e^{\gamma_c x} + A_3 R_{i_\pi} Z_{2_\pi} e^{-\gamma_\pi x} - A_4 R_{i_\pi} Z_{2_\pi} e^{\gamma_\pi x} \quad (16b)$$

where Z_{1_c} and Z_{2_c} are the characteristic impedances for the in-phase modes on lines *1* and *2*, respectively, while Z_{1_π} and Z_{2_π} are the characteristic impedances for the anti-phase modes on lines *1* and *2*, respectively. In the case where (15a) and (15b) are substituted into (1), the resultant expressions for the impedances are given by:

$$Z_{1_{c,\pi}} = \frac{z_{11} + z_{12} R_{i_{c,\pi}}}{\gamma_{c,\pi}} \qquad Z_{2_{c,\pi}} = \frac{z_{22} + z_{21} R^{-1}_{i_{c,\pi}}}{\gamma_{c,\pi}} \quad (17)$$

In the case where (15a) and (15b) are substituted into (2), the expressions for the impedances are given by:

$$Z_{1_{c,\pi}} = \frac{\gamma_{c,\pi}\left(y_{22} - y_{12} R_{i_{c,\pi}}\right)}{y_{11} y_{22} - y_{12} y_{21}} \qquad Z_{2_{c,\pi}} = \frac{\gamma_{c,\pi}\left(y_{11} - y_{21} R^{-1}_{i_{c,\pi}}\right)}{y_{11} y_{22} - y_{12} y_{21}} \quad (18)$$

Obviously these expressions for the characteristic impedances must not only be equivalent but must also be equivalent to the reciprocals of the expressions for characteristic admittances given by (9) and (10). This also can be shown through straightforward algebraic manipulation. It should be noted again, however that (17) are somewhat simpler expressions than (18), and convey more physical insight into the dependence of the coupling on the line parameters.

Thus, not only has the more general case of anisotropic coupling been treated, it has been shown that there is a duality in the forms of the expressions for the characteristic impedances and admittances depending on the initial choice of the formulation of the problem. Although all formulations must be equivalent, it has been shown that the approach taken in the development of the equations for the characteristic impedances and admittances can yield expression which more clearly relate the characteristic impedances and admittances to the circuit elements of the coupled lines.

III. Coupled-Line Four Port Circuit

A general coupled line four-port is generally described by an impedance or admittance matrix. If it is assumed that a uniform coupled-line section has a length ℓ, then the terminal voltages and currents as shown in Fig. 1 are related by the four port admittance matrix which follows from (7) and (8) and is defined by:

$$\boldsymbol{I} = \boldsymbol{Y}\boldsymbol{V} \quad (19)$$

where

$$
\mathbf{Y} = \frac{1}{R_{v_c} - R_{v_\pi}} \begin{bmatrix}
-Y_{1_c} R_{v_\pi} \coth\gamma_c \ell + Y_{1_\pi} R_{v_c} \coth\gamma_\pi \ell & Y_{1_c} \coth\gamma_c \ell - Y_{1_\pi} \coth\gamma_\pi \ell & -Y_{1_c} \operatorname{csch}\gamma_c \ell + Y_{1_\pi} \operatorname{csch}\gamma_\pi \ell & Y_{1_c} R_{v_\pi} \operatorname{csch}\gamma_c \ell - Y_{1_\pi} R_{v_c} \operatorname{csch}\gamma_\pi \ell \\
-Y_{2_c} R_{v_c} R_{v_\pi} \coth\gamma_c \ell + Y_{2_\pi} R_{v_c} R_{v_\pi} \coth\gamma_\pi \ell & Y_{2_c} R_{v_c} \coth\gamma_c \ell - Y_{2_\pi} R_{v_\pi} \coth\gamma_\pi \ell & -Y_{2_c} R_{v_c} \operatorname{csch}\gamma_c \ell + Y_{2_\pi} R_{v_\pi} \operatorname{csch}\gamma_\pi \ell & Y_{2_c} R_{v_c} R_{v_\pi} \operatorname{csch}\gamma_c \ell - Y_{2_\pi} R_{v_\pi} R_{v_c} \operatorname{csch}\gamma_\pi \ell \\
Y_{2_c} R_{v_c} R_{v_\pi} \operatorname{csch}\gamma_c \ell - Y_{2_\pi} R_{v_c} R_{v_\pi} \operatorname{csch}\gamma_\pi \ell & -Y_{2_c} R_{v_c} \operatorname{csch}\gamma_c \ell + Y_{2_\pi} R_{v_\pi} \operatorname{csch}\gamma_\pi \ell & Y_{2_c} R_{v_c} \coth\gamma_c \ell - Y_{2_\pi} R_{v_\pi} \coth\gamma_\pi \ell & -Y_{2_c} R_{v_c} R_{v_\pi} \coth\gamma_c \ell + Y_{2_\pi} R_{v_\pi} R_{v_c} \coth\gamma_\pi \ell \\
Y_{1_c} R_{v_\pi} \operatorname{csch}\gamma_c \ell - Y_{1_\pi} R_{v_c} \operatorname{csch}\gamma_\pi \ell & -Y_{1_c} \operatorname{csch}\gamma_c \ell + Y_{1_\pi} \operatorname{csch}\gamma_\pi \ell & Y_{1_c} \coth\gamma_c \ell - Y_{1_\pi} \coth\gamma_\pi \ell & -Y_{1_c} R_{v_\pi} \coth\gamma_c \ell + Y_{1_\pi} R_{v_c} \coth\gamma_\pi \ell
\end{bmatrix} \tag{20}
$$

It should be noted that, unlike for the case of bilateral coupling derived by Tripathi [2], the four-port admittance is not symmetric. This is not unexpected since the coupling mechanism is assumed to be anisotropic.

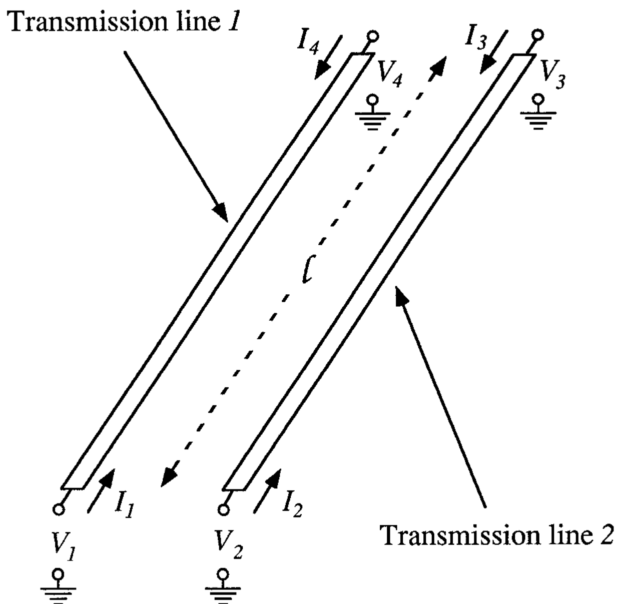

Figure 1. Generic representation of two coupled transmission lines showing the voltage and current conventions for the derivation of the four-port admittance matrix.

IV. Special Cases

The formulation presented above is applicable to a large class of problems. However, there are several significant special cases where substantial simplification of the expressions for the characteristic parameters exists. The first of these cases is that of bilateral coupling. In this case it can be shown that the characteristic admittances of the modes on the lines reduce to the expressions of Tripathi. The other special cases occur when only mutual impedances or mutual admittances exist to provide the coupling mechanism between the two lines.

A. Bilateral Coupling

Bilateral coupling between the two lines exists when the coupling mechanism is isotropic or reciprocal. This implies that the mutual impedances are equal ($z_{12} = z_{21} = z_m$) and that the mutual admittances are equal ($y_{12} = y_{21} = y_m$). By inspection it can be seen that (9) reduces to the expressions derived by Tripathi when the mutual admittances and impedances are equal. From (6), (14), and the definitions of the mode characteristic admittances it can be shown that:

$$Y_{1_{c,\pi}} = -Y_{2_{c,\pi}} R_{v_c} R_{v_\pi} \frac{b_{12}}{a_{21}} \frac{\gamma_{c,\pi}^2 - a_{11}}{\gamma_{c,\pi}^2 - b_{11}} \tag{21}$$

When the coupling is bilateral, $a_{21} = b_{12}$ and $a_{11} = b_{11}$ and hence:

$$Y_{1_{c,\pi}} = -Y_{2_{c,\pi}} R_{v_c} R_{v_\pi} \tag{22}$$

By inspection it can be seen that the four-port circuit admittance matrix is not only symmetric under these conditions but is also identical to the matrix derived by Tripathi.

B. Coupling Via Mutual Impedances Only

Under conditions where the coupling between transmission lines can be considered to occur only through the mutual impedances, the expressions for the characteristic admittances can be greatly simplified. The expressions given by (10) or (18) yield the simplest form with the characteristic admittances reducing to the self-admittance divided by the propagation constant:

$$Y_{i_{c,\pi}} = \frac{y_{ii}}{\gamma_{c,\pi}} \tag{23}$$

This situation includes the passive case where the coupling between lines can be considered to occur only through the magnetic field. In the isotropic passive case it is appropriate to think of the coupling circuit element as a mutual inductance between the two transmission lines.

C. Coupling Via Mutual Admittances Only

Under conditions where the coupling between transmission lines can be considered to occur only through the mutual admittances, the expressions for the characteristic admittances can be greatly simplified. The expressions given by (17) or (9) yield the simplest form with the characteristic admittances reducing to the propagation constant divided by the self-impedance:

$$Y_{i_{c,\pi}} = \frac{\gamma_{c,\pi}}{z_{ii}} \tag{24}$$

This situation includes the passive case where the coupling between lines can be considered to occur only through the electric field. In the isotropic passive case it is appropriate to think of the coupling circuit element as a mutual capacitance between the two transmission lines.

D. Distributed Amplifier

Distributed amplifiers are coupled transmission lines with anisotropic coupling provided by an active device such as a MESFET. Typical circuit models for distributed amplifiers employ the transistor y-parameters to model the coupling between the gate and drain lines [6,7]. In most cases the lengths of the gate and drain lines are not equal which necessitates a rescaling of the line parameters [7]. The relationships between the usual distributed amplifier model and the parameters of (1) and (2) are given by:

$$z_{11} = z_1 \tag{25}$$

$$z_{12} = z_{21} = 0 \tag{26}$$

$$z_{22} = \frac{d_2}{d_1} z_2 \tag{27}$$

$$y_{11} = y_1 + y_{11} \tag{28}$$

$$y_{12} = y_{12} \tag{29}$$

$$y_{21} = \frac{d_2}{d_1} y_{21} \tag{30}$$

$$y_{22} = \frac{d_2}{d_1} (y_2 + y_{22}) \tag{31}$$

where z_1, z_2 and y_1, y_2 are the characteristic impedances of the gate and drain lines, respectively. The per-unit length y-parameters of the MESFET coupling network are given by y_{11}, y_{12}, y_{21}, and y_{22} and are related to the active device small signal equivalent circuit [7]. The scaling factor d_2 / d_1 is the ratio of the length of the drain transmission line to the gate transmission line. An examination of (25) through (31) reveals that the coupling, assuming this model of an active device, is a special case of C as discussed above. Hence the characteristic admittances can be found from (24).

V. Conclusions

In this paper the expressions were derived for the propagation constants and characteristic admittances and impedances of general asymmetric coupled lines with anisotropic coupling. Similar to the case of isotropic coupling, it was shown that the two modes correspond to linear combinations of the voltages and currents on the lines. The anisotropic case includes some situations involving nonreciprocal material such as ferrites and also addresses the situation of distributed active devices such as field effect or junction transistors as used in distributed amplifiers. It was shown that, in contrast to previously published results, expressions for the line impedances and admittances can be formulated, which not only describe the more general anisotropic coupling case but have a simplified algebraic form which follows naturally from the choice of solving either a set of coupled voltage equations or a set of coupled current equations.

In cases where the coupling can be considered to be due to only mutual impedances or mutual admittances, these expressions were further simplified and led to a very intuitive description of the characteristic coupled-line parameters. Unlike the case where the coupling is bilateral, the four-port admittance matrix, as expected, no longer exhibits the symmetry of the bilateral-coupling case.

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